A COMBINATORIAL LI-YAU INEQUALITY AND RATIONAL POINTS ON CURVES

GUNTHER CORNELISSEN, FUMIHARU KATO, AND JANNE KOOL

Abstract. We present a method to control gonality of nonarchimedean curves based on graph theory.

Let K denote the fraction field of an excellent discrete valuation ring R. We first prove a lower bound for the gonality of a curve over the algebraic closure of K in terms of the minimal degree of a class of graph maps, namely: one should minimize over all so-called pseudo-harmonic graph homomorphisms to trees, that originate from any *refinement* of the dual graph of the stable model of the curve.

Next comes our main result: we prove a lower bound for the degree of such a graph homomorphism in terms of the first eigenvalue of the Laplacian and some "volume" of the original graph; this can be seen as a substitute for graphs of the Li-Yau inequality from differential geometry, although we also prove that the strict analogue of this conjecture fails for general graphs.

Finally, we apply the results to give a lower bound for the gonality of arbitrary Drinfeld modular curves over finite fields and for general congruence subgroups Γ of $\Gamma(1)$ that is linear in the index $[\Gamma(1):\Gamma]$, with a constant that only depends on the residue field degree and the degree of the chosen "infinite" place. This is a function field analogue of a theorem of Abramovich for classical modular curves. We present applications to uniform boundedness of torsion of rank two Drinfeld modules that improve upon existing results, and to lower bounds on the modular degree of certain elliptic curves over function fields that solve a problem of Papikian.

Contents

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1. Introduction

Abramovich **[2]** has proven a lower bound on the gonality of modular curves for congruence groups that is linear in the genus of the curves (or, what is the same, linear in the index of the group in the full modular group). His proof is an interesting combination of a lower bound on the gonality in terms of hyperbolic volume and the first eigenvalue of the Laplacian, established by Li and Yau **[29]**, together with a lower bound on this eigenvalue in the aritheoremetic case, of which a sharp value is given by Selberg's $1/4$-conjecture; the current record seems to be at $975/4096 \approx 0.238037$, due to Kim and Sarnak **[27]**. In this paper, we study a nonarchimedean analogue of the result, or rather, of the *method* of Abramovich. The main intermediate results, i.e., a lower bound on the gonality of a curve in terms of gonality of intersection dual graphs of suitable semi-stable models and an analogue of the Li-Yau result for graphs, seem to be of independent interest.

The first result is an inequality between the gonality of a curve X over a nonarchimedean valued field K (the field of fractions of an excellent discrete valuation ring R) and the "gonality" of the reduction graphs of suitable models of the curve. There are various complications, such as to establish a good theory for the reduction of a covering map $X \to \mathbf{P}^1$. Such a map extends to the stable model, but not necessarily as a *finite* morphism. This can be remedied by choosing suitable semi-stable models. The problem was studied by Liu and Lorenzini **[31]**, Coleman **[14]** and Liu **[30]**. In Section 2, we establish a slightly different version of these results, adapted to the applications we have in mind.

Next, we relate the gonality of the special fiber to what we call the *stable gonality* of the intersection dual graph. For standard graph terminology, we refer to Section 3. We also need the notion of an (indexed) pseudo-harmonic graph homomorphism, for which we refer to Definition 3.6. Given a graph G, then another graph G' is called a *refinement* of G if it can be obtained from G by performing subsequently finitely many times one of the two following operations: (a) subdivision of an edge; (b) addition of a *leaf*, i.e., the addition of an extra vertex and an edge between this vertex and a vertex of the already existing graph. The *stable gonality* of G, denoted $\mathrm{sgon}(G)$, is defined as the minimal degree of a pseudo-harmonic homomorphism from any refinement of G to a tree. This relates to, but is different from previous notions of gonality for graphs as introduced by Baker, Norine, **[6]** and Caporaso **[10]**.

1.1. Theorem. *Let X be a geometrically connected projective smooth curve over K, and $\mathscr{X}$ the stable R-model of X. Let $\overline{K}$ be an algebraic closure of K. Let $\Delta(\mathscr{X}_0)$ denote the intersection dual graph of the special fiber $\mathscr{X}_0$. Then we have*

$$\mathrm{gon}(X_{\overline{K}}) \geq \mathrm{sgon}(\Delta(\mathscr{X}_0)).$$

Two examples (3.9 and 3.10) illustrate that both refinement operation are necessary. First, the "banana graph" B_n given by two vertices joined by $n > 1$ distinct edges has stable gonality 2,

although the minimal degree of a pseudo-harmonic graph homomorphism from B_n itself (without any refinement) to a tree is n. Secondly, the minimal degree of a pseudo-harmonic graph homomorphism from the complete graph K_4 to a tree is 4. However, by adding leaves, the stable gonality can be shown to be 3.

We then prove an analogue for graphs of the theorem of Kleiman-Laksov **[28]** for the gonality of curves over arbitrary fields:

1.2. Theorem. *For any graph G with first Betti number $g \geq 2$, we have an upper bound*

$$\mathrm{sgon}(G) \leq \lfloor \frac{g+3}{2} \rfloor.$$

For this, we check that stable gonality of graphs with first Betti number ≥ 2 is defined on equivalence classes given by refinements, and use the previous theorem.

The second main result is a spectral lower bound for the stable gonality of a graph. Let λ_G denote the first non-trivial eigenvalue of the Laplacian L_G of G, and let

$$\Delta_G := \max\{\deg(v) \colon v \in V(G)\}$$

denote the maximal vertex degree of G. Finally, let $|G|$ denote the number of vertices of G. Then we have

1.3. Theorem. *The stable gonality of a graph G satisfies*

$$\mathrm{sgon}(G) \geq \left\lceil \max_{c > \Delta_G} \left\{ \frac{1}{c} \min\{\frac{1}{2}\lambda_G, c-1\} \right\} \cdot |G| \right\rceil.$$

An attractive feature of the formula is that the lower bound depends on spectral data for the original graph, not on all possible refinements of the graph. Also, since stable gonality is defined up to equivalence, one may put on the right hand side of this bound the data corresponding to any graph in the same equivalence class as G.

A similar result can be proven using the normalized graph Laplacian, replacing $|G|$ by the "volume" of the graph, cf. Theorem 7.9. For the banana graph, this lower bound is 2, and for the complete bipartite graph $K_{n,n}$, the bound is n.

The result can be seen as an analogue of the Li-Yau inequality in differential geometry **[29]**, which states that the gonality $\mathrm{gon}(X)$ of a compact Riemann surface X (minimal degree of a conformal mapping from X to the Riemann sphere) is bounded below by

$$\mathrm{gon}(X) \geq \frac{1}{8\pi} \lambda_X \mathrm{vol}(X),$$

where λ_X is the first non-trivial eigenvalue of the Laplace-Beltrami operator of X, and $\mathrm{vol}(X)$ denotes the volume of X. We will show that the strict graph theory analog of such a formula fails (7.10).

We then apply the two theorems above to Drinfeld modular curves over a general global function field K over a finite field with q elements, and we find the positive characteristic analogue of Abramovich's result. In the applications, we will write $|\mathfrak{n}|_\infty$ for the valuation corresponding to a fixed "infinite" place ∞ of degree δ of K, we denote by A the subring of K of elements that are regular outside ∞, and we let Y denote a rank-two A-lattice in the completion K_∞ of K at

∞. Up to equivalence, such lattice correspond to elements of $\mathrm{Pic}(A)$. Let H denote the maximal abelian extension of K inside K_∞; then $\mathrm{Gal}(H/K) \cong \mathrm{Pic}(A)$. In the "standard" example where $K = \mathbf{F}_q(T)$ is the function field of $\mathbf{P}^1$ and $\infty = T^{-1}$, $Y = A \oplus A$ is unique up to equivalence, and $H = K$.

1.4. Theorem. *Let Γ denote a congruence subgroup of $\Gamma(Y) := \mathrm{GL}(Y)$. Then the gonality of the corresponding Drinfeld modular curve X_Γ satisfies*

$$\mathrm{gon}_{\overline{K}}(X_\Gamma) \geq c_{q,\delta} \cdot [\Gamma(Y) : \Gamma]$$

where the constant $c_{q,\delta}$ is

$$c_{q,\delta} := \frac{q^\delta - 2\sqrt{q^\delta}}{2(q^\delta+2)(q^2-1)(q-1)}.$$

This implies a linear lower bound in the genus of modular curves of the form

$$\mathrm{gon}_{\overline{K}}(X_\Gamma) \geq c_{K,\delta} \cdot (g(X_\Gamma) - 1),$$

where $c_{K,\delta}$ is a bound that depends only on the function field K. If K is a rational function field and $\delta = 1$, then we can put $c_{K,\delta} = 2c_{q,1}$.

In the proof, we use the structure of the reduction graph of the principal modular curve of level $\mathfrak{n}$ (or rather, its components $X(Y, \mathfrak{n})$ indexed by Y running through classes in $\mathrm{Pic}(A)$). Also used in the proof is a bound of the Laplace eigenvalue for this graph that follows from the Ramanujan conjecture, proven by Drinfeld (in combination with the Courant-Weyl inequalities). The proof of the genus bound is not entirely automatic, due to possible wild ramification. The constant $c_{q,\delta}$ is probably not optimal, and it would be interesting to know whether it can be replaced by an absolute constant, or at least a constant depending on q, but tending to an absolute non-zero constant as q increases. Also notice that the bound is vacuous if $q^\delta < 4$, and that the general upper bound $(g+3)/2$ implies that any suitable constant $c_{K,\delta}$ should be smaller than $5/2$.

All previously known results on gonality of Drinfeld modular curves used point counting arguments modulo primes, rather than the above "geometric analysis" method. The best previously known bounds, due to Andreas Schweizer (**[43]**, Thm. 2.4) are not linear in the index and are only established for a rational function field $K = \mathbf{F}_q(T)$. An extra asset of the new method is that it works without much extra effort for a general function field K, rather than just a rational function field.

A first application arises from the modularity of elliptic curves over function fields. Recall that any elliptic curve E/K with split multiplicative reduction at the infinite place ∞ admits a modular parametrization $\phi\colon X_0(Y, \mathfrak{n}) \to E$ **[25]** for some suitable Drinfeld modular curve $X_0(Y, \mathfrak{n})$. This parametrization is defined over H.

1.5. Theorem. *Let E/K denote an elliptic curve with split multiplicative reduction at the place ∞, of conductor $\mathfrak{n} \cdot \infty$. Then the degree of a modular parametrization $\phi\colon X_0(Y, \mathfrak{n}) \to E$ is bounded below by*

$$\deg \phi \geq \frac{1}{2} c_{q,\delta} [\Gamma(1) : \Gamma_0(Y, \mathfrak{n})].$$

In particular, we have

$$\deg \phi \gg_{q,\delta} |\,\mathfrak{n}\,|_\infty.$$

As usual, $X \gg_y Z$ means that there exists a constant C_y depending only on y such that $X \geq C_y Z$.

For $K = \mathbf{F}_q(T)$ a rational function field, the final statement of the theorem confirms a conjecture of Papikian **[36]**, who had proven (using Spziro's conjecture for function fields and estimating symmetric square L-functions by the Ramanujan conjecture) that

$$\deg_{ns}(j_E) \cdot \deg \phi \gg_{q,\varepsilon} |\,\mathfrak{n}\,|_\infty^{1-\varepsilon},$$

where j_E is the j-invariant of E and $\deg_{ns}(j_E)$ is its inseparability degree. Actually, since he has also proven an upper bound (viz., the degree conjecture), we conclude that if E is a *strong* Weil curve over $\mathbf{F}_q(T)$, then

$$|\,\mathfrak{n}\,|_\infty \ll_q \deg \phi \ll_{q,\varepsilon} |\,\mathfrak{n}\,|_\infty^{1+\varepsilon}$$

for any $\varepsilon > 0$. In this case, contrary to the case of elliptic curves over $\mathbf{Q}$, Gekeler has proven that the modular degree always equals the congruence number of the associated automorphic form **[23]** **[13]**. Hence these results also hold for the congruence number.

We then give applications to rational points of bounded degree on various modular curves.

1.6. Theorem. *With the same notations as in Theorem 1.4, if X_Γ is defined over a finite extension K_Γ of K, such that $X_\Gamma(K_\Gamma) \neq \emptyset$, then the set*

$$\bigcup_{[L:K_\Gamma] \leq \frac{1}{2}\left(c_{q,\delta}\cdot[\Gamma(1):\Gamma]-1\right)} X_\Gamma(L)$$

is finite.

Applications to uniform bounds on isogenies and torsion points follow by applying an analogue of a method of Abramovich and Harris **[3]** and Frey **[21]**. Recall that H is the maximal abelian extension of K inside K_∞.

1.7. Corollary. *If $\mathfrak{p}$ is a prime ideal in A, then the set of all rank two Drinfeld A-modules defined over some field extension L of K that satisfies the degree bound*

$$[LH : H] \leq \frac{1}{2} c_{q,\delta} \cdot |\,\mathfrak{p}\,|_\infty$$

that admit an L-rational $\mathfrak{p}$-isogeny is finite.

We also deduce the following analogue of a result of Kamieny and Mazur **[26]**:

1.8. Corollary. *Fix a prime $\mathfrak{p}$ of A. There is a uniform bound on the size of the $\mathfrak{p}$-primary torsion of any rank two A-Drinfeld module over L, where L ranges over all extensions for which the degree $[LH : H]$ is bounded by a given constant.*

This implies that the uniform boundedness conjecture for rank-two A-Drinfeld modules over K follows from the following statement: for fixed d, there are only finitely many $\mathfrak{p}$ such that there exists an L-rational $\mathfrak{p}$-torsion point on an A-Drinfeld module over L with $[L : K] \leq d$.

For a rational function field $K = \mathbf{F}_q(T)$, the above two corollaries were proven by Schweizer **[42]**. The finite bound from these two results is not effective in the *number* of rational points. For effective results on the number of points of low degree on some Drinfeld modular curves, see for example Armana **[4]**. No analogue of Merel's theorem (uniform boundedness of torsion) is currently known for rank-two Drinfeld modules (compare also Poonen **[39]**).

As a final remark, there has recently been a surge in the use of gonality and graph theory in arithmetic, but mainly in characteristic zero; for example in the work of Ellenberg, Hall and Kowalski on generic large Galois image, coupling gonality to expander properties of Cayley graphs embedded in Riemann surfaces **[19]**. Also in our applications, in a rather different way, the graph expansion properties of the reduction graphs of Drinfeld modular curves seem to intervene in a crucial way in establishing interesting lower bound for their gonality (originally, over rational function fields, we deduced the bounds from natural bounded concentrator properties of subgraphs, as in the work of Morgenstern **[32]**).

2. Extension of covering maps

2.1. Let R be an excellent discrete valuation ring with uniformizer π, $K = \mathrm{Frac}(R)$ its field of fractions, and $k = R/\pi R$ its residue field, of characteristic $p \geq 0$. For any R-scheme $\mathscr{X}$ we denote by $\mathscr{X}_\eta$ (respectively $\mathscr{X}_0$) the generic fiber (respectively the closed fiber). We denote by $\mathscr{X}_{\mathrm{sing}}$ the singular locus of $\mathscr{X}$.

2.2. Let X be a geometrically connected projective smooth curve over K. An R-*model* of X is a pair $\mathscr{X} = (\mathscr{X}, \phi)$ consisting of an integral normal scheme $\mathscr{X}$ that is projective and flat over R and a K-isomorphism $\phi\colon \mathscr{X}_\eta \xrightarrow{\sim} X$. An R-model $\mathscr{X}$ of X is said to be *semi-stable* if its special fiber $\mathscr{X}_0$ is reduced with only ordinary double points as singularities. Such a model is called *stable* if any irreducible component of the special fiber has a finite automorphism group as a marked curve, where the marking is given by its intersection points with other components.

2.3. As was shown by Liu and Lorenzini in **[31]**, every finite morphism $f\colon X \to Y$ between geometrically connected projective smooth curves over K extends to a morphism between the stable models of X and Y, but the resulting map is *not necessarily finite*. Similar problems were already encountered and studied by Abhyankar in **[1]**. This problem occurs "in nature": for example, it follows from the results in Edixhoven's thesis **[18]** that the natural map of modular curves $X_0(p^2) \to X_0(p)$ cannot be extended to a finite morphism of stable models. However, there exists a semi-stable model admitting an extension of the map that is a finite morphism, as was shown by Coleman **[14]** and Liu **[30]**. We need a slightly different statement, that we prove along similar lines as Liu:

2.4. Theorem. *Let $f\colon X \to Y$ be a finite morphism between geometrically connected projective smooth curves over K, and $\mathscr{X}$ an R-model of X. Then there exist a finite separable field extension K'/K, semi-stable R'-models $\mathscr{X}'$ and $\mathscr{Y}'$ of $X_{K'}$ and $Y_{K'}$, respectively, over the integral closure R' of R in K', and an R'-morphism $\varphi\colon \mathscr{X}' \to \mathscr{Y}'$ such that the following conditions are satisfied*:

(a) *$\mathscr{X}'$ dominates $\mathscr{X}_{R'}$*;
(b) *φ is finite, surjective, and extends $f_{K'}$*;
(c) *the induced morphism $\varphi_0\colon \mathscr{X}'_0 \to \mathscr{Y}'_0$ satisfies $\varphi_0^{-1}((\mathscr{Y}'_0)_{\mathrm{sing}}) = (\mathscr{X}'_0)_{\mathrm{sing}}$.*

Proof. The proof is a slight modification of the proof of Proposition 3.8 in **[30]**. We first prove the theorem in the special case where f is a finite Galois covering. Let G be the Galois group of f. Then, replacing K by a finite separable extension if necessary, X has a semi-stable model $\mathscr{X}''$ that dominates $\mathscr{X}$ and admits an extension of the G-action (see Corollary 2.5 in **[30]**). We want to modify this to a semi-stable model with *inversion-free* action, as follows. Suppose an element $\sigma \in G$ of order two interchanges two components C_1 and C_2 (possibly $C_1 = C_2$) intersecting at a node u. Then we blow-up $\mathscr{X}''$ at the closed point u; we do this at all such nodes. The exceptional curves have multiplicity two. Then we replace K by a ramified quadratic extension K', and take the normalization to obtain a model $\mathscr{X}'$ of $X_{K'}$; it is clear that the G-action extends to $\mathscr{X}'$. The quotient $\mathscr{Y}' = \mathscr{X}'/G$ is a semi-stable model of $Y_{K'}$ (see Proposition 1.6 in **[31]**), and the quotient map $\varphi\colon \mathscr{X}' \to \mathscr{Y}'$ has the desired properties; we postpone the verification of property (c).

Next, we treat the case where f is separable. Let $\widetilde{X}$ denote the Galois closure of $f\colon X \to Y$. Then, replacing K by a finite separable extension if necessary, we may assume that $\widetilde{X}$ is smooth over K. As in the proof of Proposition 3.8 in **[30]**, replacing K furthermore by a finite separable extension if necessary, one has a semi-stable model $\widetilde{\mathscr{X}}$ of $\widetilde{X}$ that dominates $\mathscr{X}$ and admits an extension of the action of $G = \mathrm{Gal}(\widetilde{X}/Y)$. As in the first part, we modify $\widetilde{\mathscr{X}}$ to an inversion-free semi-stable model $\widetilde{\mathscr{X}'}$ (after replacing K by a finite separable extension). Then the obvious map

$$\varphi\colon \mathscr{X}' = \widetilde{\mathscr{X}'}/H \to \mathscr{Y}' = \widetilde{\mathscr{X}'}/G,$$

where $H = \mathrm{Gal}(\widetilde{X}/X)$, gives the desired model of f, as we will see soon below.

In general, we decompose f into a finite separable $X \to Z$ followed by a purely inseparable Frobenius map $Z \to Y \cong Z^{(p^r)}$(see Proposition 3.5 in **[30]**). The first part $X \to Z$ of the decomposition has an R'-model $\mathscr{X}' \to \mathscr{Z}'$ obtained as above. Setting $\mathscr{Y}' = \mathscr{Z}'^{(p^r)}$, we find that the composite map

$$\varphi\colon \mathscr{X}' \to \mathscr{Z}' \to \mathscr{Y}'$$

gives the answer.

The R'-morphism $\varphi\colon \mathscr{X}' \to \mathscr{Y}'$ thus obtained has properties (a) and (b). In order to show that (c) holds, it suffices to show that neither of the following two situations occurs:

(i) there exists a double point u of $\mathscr{X}'_0$ that is mapped to a smooth point of $\mathscr{Y}'_0$;
(ii) there exists a smooth point u of $\mathscr{X}'_0$ that is mapped to a double point of $\mathscr{Y}'_0$.

One can see from the construction (due to the 'inversion-free' nature) above that the situation (i) does not occur. Finally, situation (ii) is also excluded due to Proposition 1.6 in **[31]**. $\square$

3. Graphs and their stable gonality

3.1. Let G be a finite graph. In this paper, a graph can have multiple edges (this is sometimes called a "multigraph", but we will not use this terminology). We denote the sets of vertices and edges by $\mathrm{V} = \mathrm{V}(G)$ and $\mathrm{E} = \mathrm{E}(G)$, respectively. By $\mathrm{E}(x, y)$ we denote the set of edges connecting two vertices $x, y \in \mathrm{V}(G)$, and more generally, for two subsets $A, B \subseteq \mathrm{V}(G)$, we denote by $\mathrm{E}(A, B)$ the set of edges in G that connect elements from A to elements from B:

$$\mathrm{E}(A, B) = \bigcup_{x \in A \wedge y \in B} \mathrm{E}(x, y).$$

Our graphs are, unless clearly indicated, undirected, i.e., $\mathrm{E}(x,y) = \mathrm{E}(y,x)$. In case we have an oriented edge we will write (v,w) for an edge with source v and target w. The set of edges incident to a given vertex x is denoted by E_x. The cardinality of E_x, written d_x, is called the *degree* or *valancy* of v. A graph is called *k-regular* if $d_x = k$ for all $x \in \mathrm{V}$. A graph is called *loopless* if $|\mathrm{E}(x,x)| = 0$ for all $x \in V$. Two vertices x, y are called *adjacent* if $|E(x,y)| \geq 1$, and we denote it by $x \sim y$.

Another important invariant of a graphs is the *genus*, by which we mean the first Betti number $g(G) = |\mathrm{E}| - |\mathrm{V}| + 1$. Note that this differs from another convention in graph theory in which "genus" means the minimal genus of a Riemann surface in which the graph can be embedded without self-intersection. A graph of genus is 0 is called a *tree*.

Functions $f : \mathrm{V}(G) \to \mathbf{R}$, are simply called "functions on G". These form a finite dimensional vector space, equipped with the standard inner product

$$\langle f, g \rangle = \sum_{v \in \mathrm{V}(G)} f(v)g(v).$$

3.2. Denote by $A = A_G$ the adjacency matrix of G (of which the (x,y)-entry is $|E(x,y)|$) and with $D = D_G$ the diagonal matrix with the degrees of the vertices on the diagonal. Then the Laplace operator is defined by $L = L_G = D - A$.

For any graph, L_G is a real symmetric matrix, and therefore has real eigenvalues. The function $\mathbb{1}$, defined as being identically equal to 1 on V, is an eigenfunction of L_G with eigenvalue 0. The other eigenvalues are positive. We order the eigenvalues

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 ... \leq \lambda_{n-1},$$

where n is the number of vertices of the graph. It is the first non-zero eigenvalue which is important for us; we denote it by $\lambda_G := \lambda_1$.

3.3. Sometimes, one uses the *normalized Laplacian* of G, defined as

$$L_G^{\sim} = D_G^{-1/2} L_G D_G^{1/2}$$

weighted by vertex degrees (compare Chung **[12]**).

3.4. Definition. A graph G is called *stable* if all vertices have degree at least 3. A graph G' is called a *refinement* of G if it can be obtained from G by performing subsequently finitely many times one of the two following operations:

(1) subdivision of an edge,
(2) addition of a *leaf*, i.e., the addition of an extra vertex and an edge between this vertex and a vertex of the already existing graph.

3.5. One of the main tools in this paper is the notion of harmonic morphisms of graphs as developed by Baker and Norine **[6]**, and later generalized to pseudo-harmonic (indexed) morphisms Caporaso **[10]**. Since we consider "unweighted" graphs only (in the sense of Caporaso), we list the definition of (indexed) morphism from **[10]** (Definition 2.1) restricted to unweighted graphs. Notice that our use of the words *morphism* (called "indexed morphism" in **[10]**) and *refinement* (called "stable refinement" in **[10]**) is different from that source.

3.6. Definition. Let G, G' be two loopless graphs.

(1) A *morphism between* G *and* G' (denoted by $\varphi : G \to G'$) is a map

$$\varphi\colon\ \mathrm{V}(G) \cup \mathrm{E}(G) \to \mathrm{V}(G') \cup \mathrm{E}(G')$$

such that
- $\varphi(\mathrm{V}(G)) \subset \mathrm{V}(G')$,
- for every edge $e \in \mathrm{E}(x, y)$, either $\varphi(e) \in \mathrm{E}(\varphi(x), \varphi(y))$ or $\varphi(e) \in \mathrm{V}(G')$ and $\varphi(x) = \varphi(y) = \varphi(e)$,

together with, for every $e \in \mathrm{E}(G)$, a non-negative integer $r_\varphi(e)$, the *index* of φ at e, such that $r_\varphi(e) = 0$ if and only if $\varphi(e) \in \mathrm{V}(G')$.

(2) A morphism is called *pseudo-harmonic* if for every $v \in \mathrm{V}(G)$ there exists a well-defined number, $m_\varphi(v)$, such that for every $e' \in \mathrm{E}_{\varphi(v)}(G')$ we have

$$m_\varphi(v) = \sum_{e \in \mathrm{E}_v, \varphi(e) = e'} r_\varphi(e).$$

(3) A pseudo-harmonic morphism is called *non-degenerate* if $m_\varphi(v) \geq 1$ for every $v \in \mathrm{V}(G)$.

(4) A pseudo-harmonic morphism is called *harmonic* if for every $v \in \mathrm{V}(G)$

$$\sum_{e \in \mathrm{E}_v} (r_\varphi(e) - 1) \leq 2(m_\varphi(v) - 1).$$

(5) For a pseudo-harmonic morphism the following number, which is called the *degree* of φ, is independent of $v' \in \mathrm{V}(G')$ or $e \in \mathrm{E}(G')$:

$$\deg \varphi = \sum_{v \in \varphi^{-1}(v')} m_\varphi(v) = \sum_{e \in \varphi^{-1}(e')} r_\varphi(e).$$

(6) A pseudo-harmonic morphism is called a *homomorphism* if $r_\varphi(e) \geq 1$ for every $e \in \mathrm{E}(G)$.

In particular, any homomorphism is non-degenerate, and maps vertices to vertices and edges to edges, but not edges to vertices.

From the perspective of this paper, it is natural to define the following notion of gonality (this is different from existing notions of gonality, but we will discuss these later, after first showing the usefuleness of our notion in relation to morphisms of curves in the next section).

3.7. Definition. A graph G is called *stably d-gonal* if it has a refinement that allows a degree d pseudo-harmonic *homo*morphism to a tree. The *stable gonality* of a graph G is defined to be

$$\mathrm{sgon}(G) = \min\{\deg \varphi \,|\varphi\colon G' \to T\}$$

with G' a refinement of G and φ a pseudo-harmonic homomorphism to a tree T.

3.8. Remark. Although morphisms are defined only for loopless graphs, stable gonality is defined for all graphs, as loops can be "refined away" by subdividing the loop edges.

3.9. Example. The "banana graph" B_n (see Figure 1) given by two vertices joined by $n > 1$ distinct edges is the intersection dual graph of two rational curves intersecting in n points. This example occurs in nature as the stable reduction of the modular curve $X_0(p)$ over $\mathbf{Q}_p$, where n is then the

number of supersingular elliptic curves modulo p. The minimal degree of a pseudo-harmonic graph homomorphism from B_n to a tree is n. However, if we subdivide each edge once, the resulting graph admits such a homomorphism of degree 2 to a tree, which is a vertex with n edges sticking out (by identifying the two original vertices). Hence the banana graph has stable gonality equal to 2. This is compatible with the fact that the banana graph can be the intersection dual graph of both hyperelliptic and non-hyperelliptic (if $n > 3$) curves, and these are not distinguished by all subdivisions of their reduction graph.

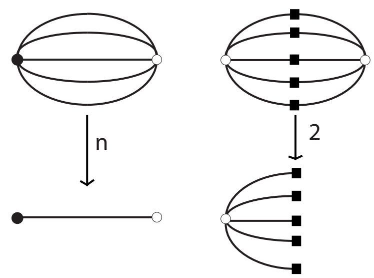


Figure 1. A banana graph B_n with a pseudo-harmonic homomorphism of (minimal) degree n, and its subdivision, with a pseudo-harmonic homomorphism of degree 2 (all indices are 1).

3.10. Example. The minimal degree of a pseudo-harmonic graph homomorphism from the complete graph K_4 to a tree is 4, but by adding leaves, such a homomorphism of degree 3 can be constructed, see Figure 2.

4. Comparing curve gonality and graph gonality: proof of Theorem 1.1

4.1. Let $\mathscr{X}$ and $\mathscr{Y}$ be R-models of geometrically connected projective smooth curves over K, and $\varphi\colon \mathscr{X} \to \mathscr{Y}$ an R-morphism. Let us say φ is *inversion-free semi-stable* if the following conditions are satisfied:

(a) $\mathscr{X}$ and $\mathscr{Y}$ are semi-stable;
(b) φ is finite and surjective;
(c) $\varphi_0^{-1}((\mathscr{Y}_0)_{\mathrm{sing}}) = (\mathscr{X}_0)_{\mathrm{sing}}$.

Theorem 2.4 says that any finite cover $f\colon X \to Y$ between geometrically connected projective smooth curves over K admits, after replacing K by a finite separable extension, an inversion-free semi-stable model f; moreover, given an arbitrary R-model $\mathscr{X}$ of X, we can take such an R-model $\varphi\colon \mathscr{X}' \to \mathscr{Y}'$ of f such that $\mathscr{X}'$ dominates $\mathscr{X}$.

4.2. Let $\varphi\colon \mathscr{X} \to \mathscr{Y}$ be an inversion-free semi-stable model of f. Consider the dual graphs $\Delta := \Delta(\mathscr{X}_0)$ and $\Gamma = \Delta(\mathscr{Y}_0)$ of the closed fibers of $\mathscr{X}_0$ and $\mathscr{Y}_0$, respectively. The vertices of Δ

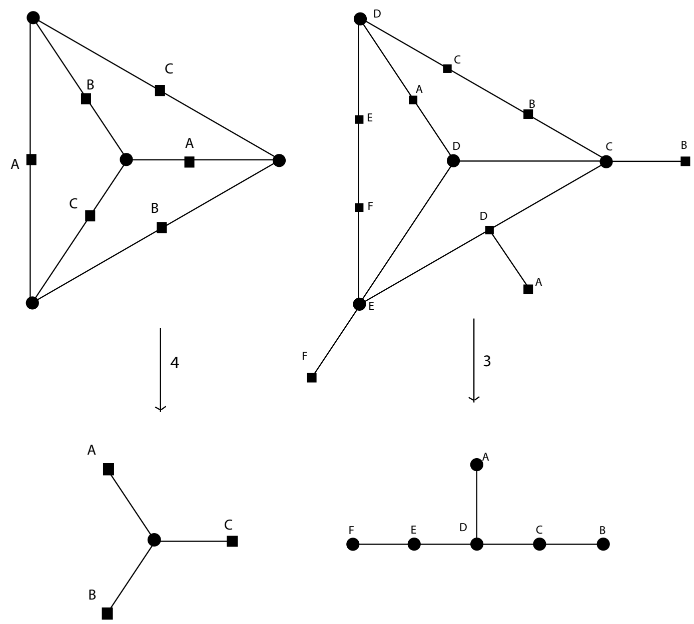


Figure 2. A subdivision of K_4 with a pseudo-harmonic homomorphism of degree 4, and a refinement (with leaves) of K_4 with a pseudo-harmonic homomorphism of degree 3 (all indices are 1).

(respectively Γ) correspond to irreducible components of $\mathscr{X}_0$ (respectively $\mathscr{Y}_0$), and two of them are connected by an edge if and only if they intersect.

The morphism φ induces the following two set-theoretic maps:

- since φ is finite, it maps each component of $\mathscr{X}_0$ surjectively onto a component of $\mathscr{Y}_0$; in particular, it induces a map $\mathrm{V}(\Delta) \to \mathrm{V}(\Gamma)$ between the sets of vertices of the graphs;
- due to the condition (c) above, each double point of $\mathscr{X}_0$ is mapped to a double point of $\mathscr{Y}_0$; that is, we have the map $\mathrm{E}(\Delta) \to \mathrm{E}(\Gamma)$ between the sets of edges.

Thus we obtain a *homomorphism* (in the sense that ϕ doesn't map edges to vertices) of graphs

$$\phi\colon \Delta \longrightarrow \Gamma.$$

4.3. We now assume that f is *separable*, and define the index r_ϕ for such f. Let $e \in \mathrm{E}(\Delta)$ be an edge with extremities $v, v' \in \mathrm{V}(\Delta)$. Let C, C' (respectively D, D') be the components of $\mathscr{X}_0$ (respectively $\mathscr{Y}_0$) corresponding to v, v' (respectively $\phi(v), \phi(v')$), respectively. The maps $C \to D$ and $C' \to D'$ ramify at the intersection point u with the same decomposition group; then define $r_\phi(e)$ to be the order of this group. This gives the indexed structure to the homomorphism ϕ of graphs in the sense of Definition 3.6.

4.4. Proposition. *For a separable $f\colon X \to Y$ that admits an inversion-free semi-stable model $\varphi\colon \mathscr{X} \to \mathscr{Y}$, the graph homomorphism $\phi\colon \Delta \to \Gamma$ constructed above is pseudo-harmonic and non-degenerate of degree* $\deg(f)$ (*in the sense of Definition 3.6*).

Proof. Let $v \in \mathrm{V}(\Delta)$ be a vertex, and C (respectively D) the component of $\mathscr{X}_0$ (respectively $\mathscr{Y}_0$) corresponding to v (respectively $\phi(v)$). Let $m_\phi(v)$ be the degree of the covering map $C \to D$. Then for any edge $e' \in \mathrm{E}(\Gamma)$ emanating from $\phi(v)$, we have

$$m_\phi(v) = \sum_{\phi(e)=e'} r_\phi(e),$$

and hence ϕ is pseudo-harmonic; moreover, it is non-degenerate since we obviously have $m_\phi(v) \geq 1$ for any $v \in \mathrm{V}(\Delta)$ (anyhow, since ϕ is a homomorphism, it is non-degenerate). $\square$

4.5. Remark. In characteristic zero, it even follows that ϕ is harmonic, by using the Riemann-Hurwitz formula as in the proof of Lemma 2.13 in **[10]**, but we will not need this stronger statement.

4.6. Corollary (=Theorem 1.1)**.** *Let X be a geometrically connected projective smooth curve over K, and $\mathscr{X}$ the stable R-model of X, and let $\Delta(\mathscr{X}_0)$ denote the intersection dual graph of the special fiber. Let $\overline{K}$ be an algebraic closure of K. Then we have*

$$\mathrm{gon}(X_{\overline{K}}) \geq \mathrm{sgon}(\Delta(\mathscr{X}_0)).$$

Proof. Gonality is the minimal degree of a map $f\colon X \to \mathbf{P}^1$. Since we work over an algebraically closed field, we can decompose a general such map into a separable part $f\colon X \to Z$ and a purely inseparable part $Z \to Z^{(p^r)} \cong \mathbf{P}^1$. Since the genus is preserved by the purely inseparable part, we find that $Z \cong \mathbf{P}^1$, too, and hence the separable part of a general map is a map of lower degree to $\mathbf{P}^1$. Hence we can restrict to bounding the degree of a separable f.

The assertion now follows from Proposition 4.4 and the following auxiliary observations.

(1) By Theorem 2.4, for any given finite cover $f\colon X \to \mathbf{P}^1_K$, replacing K by a finite separable extension, one has an inversion-free semi-stable model $\varphi\colon \mathscr{X}' \to \mathscr{P}'$ of f such that $\mathscr{X}'$ dominates $\mathscr{X}$. In particular, $\Delta(\mathscr{X}'_0)$ gives a graph that arises from $\Delta(\mathscr{X}_0)$ by subdividing some edges (corresponding to blowing up nodes) and/or adding some leaves (corresponding to blowing up smooth points) — this is exactly the notion of refinement as we have defined it.

(2) By replacing the base field K by an arbitrary finite extension K', the base-change $\mathscr{X}_{R'}$, where R' is the integral closure of R in K', is a semi-stable model of $X_{K'}$ (see Section 1.5 in **[31]**), which obviously gives the same dual graph as $\Delta(\mathscr{X}_0)$. $\square$

5. The Brill-Noether bound for stable gonality of graphs: proof of Theorem 1.2

One can use this comparison theorem to prove the analog for stable gonality of graphs of the upper bound for the gonality of curves given by Brill-Noether theory: a curve of genus g over an algebraically closed field has gonality bounded above by $\lfloor (g+3)/2 \rfloor$; this was proven in general by Kleiman and Laksov **[28]**. To prove this for graphs, we first show that pseudo-harmonic homomorphisms can be "refined", in a sense to be made precise.

5.1. Definition. Any two refinements G_1 and G_2 of a graph G have a minimal common refinement, called the *join* $G_1 \vee G_2$ of G_1 and G_2.

5.2. Definition. A refinement G' of a graph G induces refinements of all of its subgraphs. If $e \in \mathrm{E}(v, w)$ is an edge in G that connect two vertices $v, w \in \mathrm{V}(G)$, denote by $[e]$ the subgraph of G consisting of the vertices v and w joined by the edge e. Denote with $G'[e]$ the refinement of $[e]$ in G'.

5.3. Definition. A *refinement of a pseudo-harmonic homomorphism* $\varphi : G \to T$ is a pseudo-harmonic homomorphism

$$\varphi' : G' \to T'$$

such that G' (respectively T') is a refinement of G (respectively T), and such that

1. for all $v \in \mathrm{V}(G)$, $\varphi'(v) = \varphi(v)$;
2. for any $v, w \in \mathrm{V}(G)$ and any edge $e \in \mathrm{E}(v, w)$, every refinement of $[e]$ in G' is mapped to a refinement of $[\varphi(e)]$ in T', viz.,
$$\varphi(G'[e]) = T'[\varphi(e)];$$
3. for all $e' \in \mathrm{E}(G')$, the index $r_{\varphi'}(e) = 1$;
4. $\deg \varphi' = \deg \varphi$.

5.4. Lemma. *For any pseudo-harmonic homomorphism $\varphi : G \to T$ there exists a refinement.*

Proof. The refinement is constructed using the following recipe:

1. subdivide every edge in T;
2. subdivide every edge in G;
3. add to every vertex v in the original graph G, $\sum_{e \in \mathrm{E}_v} (r_\varphi(e) - 1)$ leaves;
4. asign to every edge in the new graph index 1;
5. extend φ in the obvious way.

□

5.5. Lemma. *Let $\varphi : G \to T$ be a pseudo-harmonic homomorphism and let H be a refinement of G, then there is a refinement $\varphi' : G' \to T'$ of φ, such that G' is a refinement of H.*

Proof. Using Lemma 5.4, we can find a refinement of $\varphi : G \to T$, which we denote by the same symbols. Then use the following recipe:

1. replace every edge e_0 in T by
$$\bigvee_{e \in \varphi^{-1}(e_0)} H([e])$$
and call the resulting new tree T';
2. replace every edge e in G by $T'[\varphi(e)]$;
3. extend φ in the obvious way.

□

5.6. Corollary. *Call two graphs* equivalent *if they are refinements of the same stable graph. This defines an equivalence relation on the set of all graphs of genus at least* 2*. The map* sgon *is defined on equivalence classes of graphs.*

Proof. Let G' be a refinement of G. It follows from the definition that $\mathrm{sgon}(G') \geq \mathrm{sgon}(G)$. Since refinement of morphisms preserves degree, the previous lemmas implies that the other inequality $\mathrm{sgon}(G') \leq \mathrm{sgon}(G)$ also holds. □

5.7. Theorem (= Theorem 1.2)**.** *For any graph G of genus $g \geq 2$, the Brill-Noether bound holds:*

$$\mathrm{sgon}(G) \leq \lfloor \frac{g+3}{2} \rfloor.$$

Proof. Since sgon is defined on the equivalence classes of graphs it is sufficient to prove the bound for one representative of each equivalence class. It is sufficient to show that any stable graph G of genus $g \geq 2$ admits a refinement G' such that there exists a curve X such that G' is the dual graph of the minimal model of X. Indeed, since the genus of X equals the genus of G', which equals g (since the genus of a graph doesn't change under refinement), the classical bound $\mathrm{gon}_{\overline{K}}(X) \leq (g+2)/3$ holds (cf. Kleiman-Laksov **[28]**). The result follows from $\mathrm{sgon}(G') \leq \mathrm{gon}_{\overline{K}}(X)$ (Theorem 1.1).

We now show the existence of such a refinement. Let G be a stable graph of genus $g \geq 2$ and let $\Delta = \max\{d_x | x \in \mathrm{V}(G)\}$. Choose g edges $e_1, ..., e_g$ of G such that $G - \{e_1, ..., e_g\}$ is a tree. Replace each edge e_i (connecting two vertices x_i and y_i) by two edges $[x_i, v_i]$ and $[w_i, y_i]$, where v_i and w_i are new vertices not connected to any other vertex. In this way, G is replaced by a tree T_G. Choose an embedding of T_G in the Bruhat-Tits tree $\mathcal{T}$ for $k = \mathbf{F}_q((t))$, where q satisfies $q+1 \geq \Delta$. Denote the images of (v_i, w_i) in $\mathcal{T}$ by the same letters. Now choose hyperbolic elements $\gamma_1, ..., \gamma_g$ in $\mathrm{PGL}(2, k)$ such that each γ_i acts as translation along a geodesic through v_i and w_i, and $\gamma_i(v_i) = w_i$. Then $\Gamma = \langle \gamma_1, ..., \gamma_g \rangle$ is a Schottky group. Denote by $\mathcal{T}_\Gamma$ the subtree of $\mathcal{T}$ spanned by the limit set of Γ. Then $G' \simeq \Gamma \backslash \mathcal{T}_\Gamma$, where G' is the refinement of G given by subdividing each of the edges $e_1, ..., e_g$ once. Also, G' is the intersection dual graph of the minimal model of the Mumford curve corresponding to Γ (**[33]**, page 164). □

It would be interesting to have a purely graph theoretical proof of the above result.

6. Intermezzo on different notions of gonality

The following definition of a different notion of gonality of graphs is taken from **[10]**:

6.1. Definition. A graph G is called *d-gonal* if it allows a non-degenerate harmonic morphism of degree d to a tree. The gonality of a graph is defined to be

$$\mathrm{gon}(G) = \min\{\deg \varphi | \varphi \text{ a non-degenerate harmonic morphism to a tree } T\}.$$

One should remember that this graph gonality is defined in terms of *morphisms*, not *homomorphisms*; i.e., the maps can contract edges. Also, one needs to impose the condition of non-degeneracy. The minimal degree of a morphism (with no further requirements) of a graph G to a tree is bounded above by the minimal degree of a vertex of G. Indeed, let v denote a vertex with minimal degree in G, and consider the map $\varphi \colon G \to T$ to a segment T (i.e., the tree consisting of two vertices v_1 and v_2 connecting by one edge e), given by the map

$$\begin{aligned}
&\varphi(v) = v_1; \\
&\varphi(E_v) = e; \\
&\varphi(\mathrm{V}(G) \cup \mathrm{E}(G) - (\{v\} \cup E_v)) = v_2.
\end{aligned}$$

This is in general a non-degenerate map, namely, as soon as there is a vertex w in G that is not connected to v, this vertex has $m_\varphi(w) = 0$. If we would further allow for refinements (as is required by the reduction theory of morphisms), we can always find vertices of valency 2, and then this map would always have degree 2.

The following elementary fact, a "trivial" spectral bound on the gonality, does not seems to have been observed before:

6.2. Proposition. *The gonality of a graph G is bounded below by the edge-connectivity (viz., the number of edges that need to be removed from the graph in order to disconnect it):*

$$\mathrm{gon}(G) \geq \eta(G).$$

If G is a simple graph (i.e., without multiple edges), unequal to a complete graph, then

$$\mathrm{gon}(G) \geq \lambda_G.$$

Proof. Let $\varphi\colon G \mapsto T$ denote a pseudo-harmonic non-degenerate morphism. Choose any edge $e \in \mathrm{E}(T)$. Since removing e from T disconnects it, $\varphi^{-1}(e)$ is a set of edges of G whose removal disconnects G. Hence

$$\mathrm{gon}(G) \geq |\varphi^{-1}(e)| \geq \eta(G).$$

For a simple graph which is not complete, the bound

$$\eta(G) \geq \lambda_G$$

is one of the inequalities of Fiedler **[20]** (4.1 & 4.2). □

6.3. Remark. The "trivial" spectral bound in the above proposition is not very useful in practice, since it does not contain a "volume" term (like the Li-Yau inequality). Also, since every graph acquires edge connectivity two or one by refinements, the lower bound in the proposition trivializes under refinements (which are required by the reduction theory of morphisms).

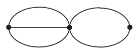
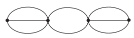
Yet another notion of gonality of graphs was introduced by Baker in **[5]**, based on linear systems on graphs — we do not recall all definitions. Following Caporaso, we call it *divisorial gonality*. There is in general no relation (inequality either side) between gonality and divisorial gonality, cf. **[10]** Remark 2.7. In **[9]**, Caporaso has proven a Brill-Noether bound for divisorial gonality.

We now compare our notion of stable gonality to these other notions of gonality, by simply computing some examples. We have collected these in Table 1, that gives the values of gonality and various other invariants of some graphs G. As above, λ_G is the first eigenvalue of L_G, and $\lambda_G^\sim$ is the first eigenvalue of the normalized Laplacian $L_G^\sim$; $\eta(G)$ is the edge connectivity, $\mathrm{vol}(G)$ is the volume of the graph (sum of degrees over all vertices; so twice the number of edges); $\mathrm{gon}(G)$ is the gonality, $\mathrm{dgon}(G)$ is the divisorial gonality, and $\mathrm{sgon}(G)$ is the stable gonality of G. We leave out the lengthy but elementary calculations (for the divisorial gonality of K_n, we refer to **[5]**, 3.3).

One observes the following:

(1) The banana graph B_n has divisorial and stable gonality 2 but edge connectivity n (cf. Table 1), showing that an equality analogous to the one in Proposition 6.2 cannot hold for divisorial or stable gonality.
(2) Stable gonality is not always equal to gonality, as the example of the banana graph shows.

Table 1. Some graphs and their invariants, including gonalities

Graph G	$\operatorname{sgon}(G)$	$\operatorname{gon}(G)$	$\operatorname{dgon}(G)$	$\eta(G)$	λ_G	$\lvert G\rvert$	$\lambda_G^{\sim}$	$\operatorname{vol}(G)$
Complete graph K_n	$n-1$	$n-1$	$n-1$	$n-1$	n	n	$\frac{n}{n-1}$	$n(n-1)$
Cycle graph C_n	2	2	2	2	$4\sin^2(\frac{\pi}{n})$	n	$2\sin^2(\frac{\pi}{n})$	$2n$
Utility graph $K_{3,3}$	3	3	3	3	3	6	1	18
Banana graph B_n	2	n	2	n	$2n$	2	2	$2n$
	3	3	3	2	$5-\sqrt{7}\approx 2.35$	3	λ_G	10
	3	3	4	2	$5-\sqrt{13}\approx 1.93$	4	$\frac{2}{5}$	16

(3) Stable gonality is not always equal to divisorial gonality, as the example on the last line shows.

One may wonder whether divisorial gonality is always larger than stable gonality.

7. A spectral lower bound for the stable gonality of a graph: Proof of Theorem 1.3

7.1. Let G be a graph, and let G' be a refinement of G. Let $\varphi\colon G' \to T$ be a pseudo-harmonic homomorphism from the refinement to a tree T. In this section we derive a Li-Yau like estimate **[29]** for the degree of φ based on data from the original graph G; we relate the degree of φ with the eigenvalue λ_G and the number of vertices of G.

7.2. The main lemma below is a Cheeger-like result relating the eigenvalue to the size of the inverse images of a splitting of the tree into two parts; namely, by removing an edge e from a tree T, one obtains a disconnected tree, i.e.,

$$T - e = T_1 \bigsqcup T_2.$$

The intersections

$$G_i := \mathrm{V}(G) \cap \varphi^{-1}(\mathrm{V}(T_i))$$

for $i = 1, 2$ form a partition of $\mathrm{V}(G)$, the vertices of the original graph.

7.3. Lemma. *Let G, G', $\varphi\colon G' \to T$ and G_1 and G_2 be as above, then*

$$\frac{1}{2}\lambda_G \min(|G_1|, |G_2|) \leq \deg(\varphi).$$

Proof. First, note that the inequality is trivial if $\min(|G_1|, |G_2|) = 0$. Now assume the minimum is non-zero. The estimate follows from the variational characterization of λ_G via the Rayleigh-quotient,

$$\lambda_G = \inf_{f \perp \mathbb{1}} \frac{\langle f, Lf\rangle}{\langle f, f\rangle} = \inf_{f\perp\mathbb{1}} \frac{\sum\limits_{u\sim v} (f(u)-f(v))^2}{\sum_v f(v)^2},$$

where notations are as in 3.1.

We construct an appropriate function f based on the pseudo-harmonic homomorphism $\varphi\colon G' \to T$ and the removed edge $e \in T$, as follows:

$$f(v) = \begin{cases} \frac{1}{|G_1|} & \text{if } v \in G_1, \\ -\frac{1}{|G_2|} & \text{if } v \in G_2. \end{cases}$$

It is easy to check that $f \perp \mathbb{1}$, and therefore,

$$\begin{aligned} \lambda_G &\leq \frac{\sum_{u \sim v}(f(u) - f(v))^2}{\sum_v f(v)^2} \\ &= |\operatorname{E}(G_1, G_2)|(\frac{1}{|G_1|} + \frac{1}{|G_2|}) \\ &\leq \frac{2|\operatorname{E}(G_1, G_2)|}{\min(|G_1|, |G_2|)} \end{aligned}$$

We finish the proof by showing that $\deg(\varphi) \geq |\operatorname{E}(G_1, G_2)|$. Suppose an edge $e \in \operatorname{E}(G_1, G_2)$ is replaced in G' by a path $p(e)$, possibly of length 1. Let us describe this path as a series of edges $e_1, \ldots, e_n \in \operatorname{E}(G')$, such that e_1 is incident to a vertex in G_1, and e_n is incident to a vertex in G_2. Then for at least one of the e_i it holds that $\varphi(e_i) = e$. The desired inequality follows. $\square$

7.4. This spectral estimate for the degree is best if $\min(|G_1|, |G_2|)$ is maximal among all possible choices for the edge $e \in T$. Let us make our notation a bit more precise. Again, let G' be a refinement of G, and let $\varphi : G' \to T$ be a pseudo-harmonic homomorphism. If an edge $e \in T$ is equipped with an orientation, denote by $G_1(e)$ (respectively, $G_2(e)$) the vertices v in the original graph G for which $\varphi(v)$ is in the connected component of the source of e (respectively, of the target of e). For a vertex $x \in T$ we denote by

$$S_x = |\operatorname{V}(G) \cap \varphi^{-1}(x)|.$$

Note that it could happen that $S_x = 0$. Our main theorem in this section states that there is a dichotomy: either the two sets $G_1(e)$ and $G_2(e)$ are relatively large, or there is a vertex for which S_x is large:

7.5. Theorem. *Let G be a graph, and let G' be a refinement of G. Let $\varphi\colon G' \to T$ be a pseudo-harmonic homomorphism of graphs and let*

$$c \geq \Delta_T = \max\{d_x | x \in V(T)\}$$

be an integer, then one of the following situations occurs:

(1) *Either there is an edge $e \in T$ such that after choosing an orientation, we have*

$$\min(|G_1(e)|, |G_2(e)|) \geq \frac{1}{c}|G|;$$

(2) *or there is a vertex $x \in T$ such that*

$$S_x \geq \frac{c - d_x}{c}|G|.$$

Proof. Suppose that for all edges e, it holds true that

$$\min(|G_1(e)|, |G_2(e)|) < |G|/c.$$

Call a vertex of degree one on T an *end* of T. Choose an end $x_0 \in V(T)$ such that x_0 has maximal S_x for all such ends x of T:

$$S_{x_0} = \max\{S_x | x \text{ an end of } T\}.$$

Define an orientation of the edges of G' such that x_0 is the only end of T such that x_0 is the source of an edge, and such that for all edges of T, x_0 is on the source side. Let W_1 be a path from x_0 to an other end of T. Label the edges in this path $e_1, \dots, e_n$, and the vertices as $x_0, \dots, x_n$, so that $e_i = (x_{i-1}, x_i)$. If

$$\min(|G_1(e_1)|, |G_2(e_1)|) = |G_2(e_1)|,$$

then by assumption $|G_2(e_1)| < |G|/c$, and hence

$$|G_1(e_1)| = |G| - |G_2(e_1)| \geq \frac{c-1}{c}|G|$$

and the situation is as in (2) for $x = x_0$, since $S_{x_0} = |G(e_1)|$. Without loss of generality, we can now assume that $|G_1(e_1)| \leq \frac{c-1}{c}|G|$.

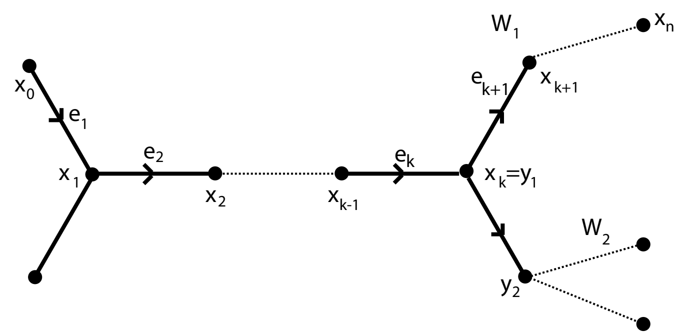


Figure 3. Illustration of the proof of Theorem 7.5

Next, suppose that the other possibility for the minimum holds: $\min(|G_1(e_1)|, |G_2(e_1)|) = |G_1(e_1)|$. Then

$$|G_1(e_1)| < |G_1(e_2)| < \cdots < |G_1(e_n)|,$$

since the number of edges on the source side increases as we move from edge e_i to edge e_{i+1}. If for the last edge $\min(|G_1(e_n)|, |G_2(e_n)|) = |G_1(e_n)|$, then

$$|G_2(e_n)| \geq \frac{c-1}{c}|G| \geq |G_1(e_1)|,$$

the last inequality being our assumption from before. Since $|G_2(e_n)| = S_{x_n}$, this is a contradiction with the assumption that x_0 attains the maximum of S_x amongst all ends x of T. Therefore, there has to be a "turning point" $1 \leq k < n$, such that

$$\begin{aligned}&\min(|G_1(e_k)|, |G_2(e_k)|) = |G_1(e_k)| \text{ and}\\&\min(|G_1(e_{k+1})|, |G_2(e_{k+1})|) = |G_2(e_{k+1})|.\end{aligned}$$

First of all, if all paths W from x_0 to another end of T that pass through $y_1 := x_k$ have their turning point at that same point y_1, then we are done, since

$$\begin{aligned}S_{y_1} &= |G_2(e_k)| - \sum_W |G_2(e_{k+1}(W))|\\&= |G| - |G_1(e_k)| - \sum_W |G_2(e_{k+1}(W))|\\&\geq |G| - \frac{1}{c}|G| - \frac{|W|}{c}|G|\\&= \frac{c - d_{y_1}}{c}|G|,\end{aligned}$$

Next, if not all such paths have their turning point at y_1, choose such a path W_2 different from W_1 through y_1, and let y_2 denote its turning point. The point y_2 will be at further distance from x_0 than y_1. If all paths from x_0 to another end of T that pass through y_2 have their turning point at that same point y_2, we are done again, as above with y_2 instead of y_1. If not, choose a path W_3 different from W_1 and W_2, and let y_3 denote its turning point, and so on. This process will terminate, since every path has a turning point strictly before reaching its end, and the tree T is finite. □

7.6. Corollary (= Theorem 1.3)**.** *The stable gonality of a graph G satisfies*

$$\mathrm{sgon}(G) \geq \left\lceil \max_{c > \Delta_G} \left\{ \frac{1}{c} \min\{\frac{1}{2}\lambda_G, c-1\} \right\} \cdot |G| \right\rceil.$$

We now give some examples that illustrate the effectivity of the bound.

7.7. Example. For the banana graph B_n, we have $\Delta_{B_n} = n$, $|B_n| = 2$ and $\lambda_{B_n} = 2n$, so the lower bound is

$$\mathrm{sgon}(B_n) \geq 2$$

for $n \geq 2$. See Figure 1 for such a map of degree 2.

7.8. Example. For the complete bipartite graph $K_{n,n}$, we have $\Delta_{K_{n,n}} = n$, $|K_{n,n}| = 2n$ and $\lambda_{K_{n,n}} = n$, so the lower bound is

$$\mathrm{sgon}(K_{n,n}) \geq n.$$

A morphism which attains this degree is given by mapping $K_{n,n}$ to the star with one central vertex and n emanating edges in the obvious way. For $n = p^r + 1$ (p prime), the graph $K_{n,n}$ occurs as stable reduction graph of the curve

$$(x^{p^r} - x)(y^{p^r} - y) = \lambda$$

(seen in $\mathbf{P}^1 \times \mathbf{P}^1$) with $|\lambda| < 1$, over a field K of characteristic p, which, as a fiber product of two projective lines, admits an obvious morphism of degree $p^r + 1$ to $\mathbf{P}^1$. The stable reduction itself consist of two transversally intersecting families of $p^r + 1$ rational curves ("check board with p^{2r} squares"). For more details on these curves, see for example **[15]**.

A similar result holds for the Laplace L_G replaced by the normalized Laplace $L_G^\sim$ (see (3.3)):

7.9. Theorem. *Denote by $\lambda_G^\sim$ the smallest nonzero eigenvalue corresponding to the normalized Laplacian $L_G^\sim$ on a graph G; then the stable gonality of G satisfies*

$$\operatorname{sgon}(G) \geq \left\lceil \max_{c > \Delta_G} \left\{ \frac{1}{c} \min\{\frac{1}{2}\lambda_G^\sim, c-1\} \right\} \cdot \operatorname{vol}(G) \right\rceil,$$

where $\operatorname{vol}(G) = \sum_{v \in G} d_v$. □

The proof of this inequality is virtually the same as the one of Theorem 1.3 and we will not give the details. Also, the bound is practically the same, since the decrease in eigenvalue seems to be compensated by an increase in volume. For example, for a k-regular graph $\lambda_G = k\lambda_G^\sim$, while $k|G| = \operatorname{vol}(G)$.

7.10. Remark. The famous Li-Yau inequality from differential geometry **[29]** states that the gonality $\operatorname{gon}(X)$ of a compact Riemann surface X (minimal degree of a conformal mapping φ of X to the Riemann sphere) is bounded below by

$$\operatorname{gon}(X) \geq \frac{1}{8\pi}\lambda_X \operatorname{vol}(X),$$

where λ_X is the first non-trivial eigenvalue of the Laplace-Beltrami operator of X, and $\operatorname{vol}(X)$ denotes the volume of X.

For graphs G with any Laplacian (normalized or not) and for all three notions of gonality (plain, divisorial and stable) an inequality of the form

$$\text{“}\operatorname{gon}(G) \geq \kappa \cdot \lambda_G \cdot \operatorname{vol}(G)\text{”} \quad (*)$$

for some constant κ fails. A counterexample is given by the complete graph K_n, which has all three gonalities equal to $n-1$. However, a lower bound of the form $(*)$ would be $\kappa \cdot n^2(n-1)$ for the usual Laplacian, and $\kappa \cdot n^2$ for the normalized Laplacian (see Table 1 for the data).

As we have seen in Corollary 5.6, stable gonality is defined on equivalence classes of graphs, in the sense that two graphs G and G' are equivalent (notation $G \sim G'$) if they are refinements of the same stable graph. Hence the result also implies that

7.11. Corollary. *For any graph G with $g \geq 2$, we have*

$$\operatorname{sgon}(G) \geq \max_{G' \sim G} \left\lceil \max_{c > \Delta_{G'}} \left\{ \frac{1}{c} \min\{\frac{1}{2}\lambda_{G'}, c-1\} \right\} \cdot |G'| \right\rceil. \ \square$$

8. A linear lower bound on the gonality of Drinfeld modular curves: proof of Theorem 1.4

We recall the main concepts and notations from the theory of general Drinfeld modular curves, cf. **[22]**, **[25]**.

8.1. Let K denote a global function field of a smooth projective curve X over a finite field $k = \mathbf{F}_q$ with q elements and characteristic $p > 0$, and ∞ a place of degree δ of K. Let A denote the subring of K of elements that are regular outside ∞.

8.2. Let Y denote a rank-two A-lattice in the completion K_∞ of K at ∞. Such lattices are classified up to isomorphism by their determinant, so they are isomorphic to $A \oplus I$, where I runs through a set of representatives of $\mathrm{Pic}(A)$, the ideal class group of A.

Let $\mathrm{GL}(Y)$ denote the automorphism group of the lattice Y:

$$\mathrm{GL}(Y) = \{\gamma \in \mathrm{GL}_2(K) \colon \gamma Y = Y\},$$

and let Γ denote a congruence subgroup of $\Gamma(Y) := \mathrm{GL}(Y)$. This means that Γ contains a principal congruence group $\Gamma(Y, \mathfrak{n})$ as a finite index subgroup, where

$$\Gamma(Y, \mathfrak{n}) = \ker\left(\Gamma(Y) \to \mathrm{GL}(Y/\,\mathfrak{n}\,Y)\right),$$

for $\mathfrak{n}$ an ideal in A. Let $Z \cong \mathbf{F}_q^*$ denote the center of $\mathrm{GL}(Y)$.

If $Y = A \oplus A$ is the "standard" lattice, we revert to the standard notations $\Gamma(1) := \Gamma(A \oplus A)$ and $\Gamma(\mathfrak{n}) := \Gamma(A \oplus A, \mathfrak{n})$.

8.3. The groups Γ act by fractional transformations on the Drinfeld space $\Omega = \mathbf{C}_\infty - K_\infty$, where $\mathbf{C}_\infty$ is the completion of an algebraic closure of K_∞. The quotient $\Gamma\backslash\Omega$ is an analytic smooth one-dimensional space, and is the analytification of a smooth affine algebraic curve Y_Γ, that can be defined over a finite abelian extension of K inside K_∞. It can be compactified to a Drinfeld modular curve X_Γ by adding finitely many points, called cusps.

The $\mathbf{C}_\infty$-points of the (coarse) moduli scheme $M(\mathfrak{n})$ of rank-two Drinfeld A-modules with full level $\mathfrak{n}$-structure (i.e., an isomorphism of $(A/\,\mathfrak{n})^2$ with the torsion of the Drinfeld module) can be described as

$$M(\mathfrak{n})(\mathbf{C}_\infty) = \bigsqcup_{Y \in \mathrm{Pic}(A)} \Gamma(Y, \mathfrak{n})\backslash\Omega.$$

We denote such a component by $Y(Y, \mathfrak{n}) := \Gamma(Y, \mathfrak{n})\backslash\Omega$, and its compactification by $X(Y, \mathfrak{n})$.

8.4. The groups Γ also act by automorphisms on the Bruhat-Tits tree $\mathcal{T}$ of $\mathrm{PGL}(2, K_\infty)$ **[44]**. The quotient $\Gamma\backslash\mathcal{T}$ is the union of a finite graph $G_\Gamma := (\Gamma\backslash\mathcal{T})^0$ and a finite number of half lines in correspondence with the cusps of X_Γ, and the curve X_Γ is a Mumford curve over K_∞ **[33]** such that the intersection dual graph of the reduction, which is a finite union of rational curves over $\mathbf{F}_{q^\delta}$ intersecting transversally in $\mathbf{F}_{q^\delta}$-rational points, equals the finite graph G_Γ.

8.5. Theorem (= Theorem 1.4)**.** *Let* Γ *denote a congruence subgroup of* $\Gamma(Y)$. *Then the gonality of the corresponding Drinfeld modular curve* X_Γ *satisfies*

$$\mathrm{gon}_{\overline{K}}(X_\Gamma) \geq c_{q,\delta} \cdot [\Gamma(Y) : \Gamma]$$

where the constant $c_{q,\delta}$ is

$$c_{q,\delta} := \frac{q^\delta - 2\sqrt{q^\delta}}{2(q^\delta+2)(q^2-1)(q-1)}.$$

This implies a linear lower bound in the genus of modular curves of the form

$$\mathrm{gon}_{\overline{K}}(X_\Gamma) \geq c_{K,\delta} \cdot (g(X_\Gamma) - 1),$$

where $c_{K,\delta}$ is a bound that depends only on the function field K. If K is a rational function field and $\delta = 1$, then we can put $c_{K,\delta} = 2c_{q,1}$.

Proof. The proof has various parts.

Reduction to principal congruence groups. First of all, we observe that it suffices to prove the bound for the groups $\Gamma(Y, \mathfrak{n})$. Indeed, if $\varphi\colon X_\Gamma \to \mathbf{P}^1$ is a morphism, then from the inclusion $\Gamma(Y, \mathfrak{n}) \leq \Gamma$ we get a composed morphism

$$X_{\Gamma(Y,\mathfrak{n})} \to X_\Gamma \to \mathbf{P}^1 \tag{1}$$

of degree

$$\frac{[\Gamma : \Gamma(Y, \mathfrak{n})]}{|\Gamma \cap Z|} \cdot \deg \varphi,$$

and hence

$$\mathrm{gon}(X_\Gamma) \geq \mathrm{gon}(X(Y, \mathfrak{n}))/[\Gamma : \Gamma(Y, \mathfrak{n})]. \tag{2}$$

Therefore, the desired inequality

$$\mathrm{gon}(X_\Gamma) \geq c_q[\Gamma(Y) : \Gamma]$$

follows from

$$\mathrm{gon}(X(Y, \mathfrak{n})) \geq c_q[\Gamma(Y) : \Gamma(Y, \mathfrak{n})].$$

We now prove the gonality bound by invoking Theorem 1.3 for the reduction graph of the Drinfeld modular curve $X(Y, \mathfrak{n})$. We first compute the necessary spectral data from the covering $G := G_{\Gamma(Y,\mathfrak{n})} \to G_{\Gamma(Y)}$.

A lower bound on the number of vertices. Both of these graphs are the finite parts of quotients of the Bruhat-Tits tree $\mathcal{T}$ of $\mathrm{PGL}(2, K_\infty)$, in which every vertex is $(q^\delta + 1)$-regular. Let us consider the special vertex of $\mathcal{T}$ corresponding to the class of the trivial rank-two vector bundle $[\mathcal{O}_\infty \oplus \mathcal{O}_\infty]$ on X, and let v_0 denote the corresponding vertex in $G_{\Gamma(Y)}$. The stabilizer of this vertex is precisely $PGL(2, \mathbf{F}_{q^\delta})$ (namely, an element of the stabilizer induces an automorphism of the "star" of the vertex, which is given by $\mathbf{P}^1(\mathbf{F}_{q^\delta})$.) The stabilizer intersects $\Gamma(Y)/Z$ (where Z is the center) in the "constant group" $\mathrm{PGL}(2, \mathbf{F}_q)$, and the group $\Gamma(Y, \mathfrak{n})$ (for $\mathfrak{n} \neq 1$) in the trivial group. We conclude that

$$|\,\mathrm{V}(G)| \geq \frac{1}{q(q^2-1)} \cdot [\Gamma(Y) : \Gamma(Y, \mathfrak{n})], \tag{3}$$

since the right hand side is the number of vertices in $\Gamma(Y, \mathfrak{n})$ above v_0, and $\mathrm{PGL}(2, \mathbf{F}_q)$ has cardinality $q(q^2 - 1)$.

8.6. Remark. This estimate for the number of vertices of $G_{\Gamma(Y,\mathfrak{n})}$ will be enough for our purposes, since it differs from the index only by a constant in q. But one might also count the total number of vertices of the graph. For a rational function field $K = \mathbf{F}_(T)$ with a place ∞ of degree one, this is easily done, the result being

$$|V(G_{\Gamma(\mathfrak{n})})| = \frac{2q^{\deg(\mathfrak{n})+1} - q - 1}{q^{\deg(\mathfrak{n})+1}(q^2-1)(q-1)}[\Gamma(1) : \Gamma(\mathfrak{n})];$$

compare also with computations in **[32]** (cf. **[11]**, **[41]**) and **[24]**. It seems another proof of the lower bound on the gonality is possible by using Morgenstern's result that there is a perfect matching between a very large (constant fraction depending only on q, not on $\deg(\mathfrak{n})$) subset of the vertices above v_0 in $G_{\Gamma(\mathfrak{n})}$ and vertices in the complement, but we did not pursue this, since it would give a less general and worse result.

8.7. Remark. The gonality is *not* always realized by the obvious map $X_\Gamma \to X(1) \cong \mathbf{P}^1$. For example, set $q = 2$ and let $\mathfrak{p}$ denote an prime of degree 3; then the modular curve $X_0(\mathfrak{p})$ is hyperelliptic, but the map $X_0(\mathfrak{p}) \to X(1)$ has degree 9. Also notice that for a general base field K, the modular curve $X(1)$ is not even itself a rational curve.

8.8. Remark. Counting the number of cusps (so the number of vertices above a vertex in $G_{\Gamma(Y)}$ corresponding to a split bundle of high degree) is not enough to get a linear estimate in the index, since the cusps have rather large stabilizers (of size roughly the third root of the index).

The maximal degree of a vertex Since the tree $\mathcal{T}$ is $(q^\delta + 1)$-valent, the maximal valency of a vertex in any graph $G_{\Gamma(Y,\mathfrak{n})}$ is bounded above by $q^\delta + 1$. Actually, this bound is attained at any vertex above v_0, since the vertex and edge stabilizers in $G = G_{\Gamma(Y,\mathfrak{n})}$ are trivial; the edge stabilizer in $\mathrm{PGL}(2, K_\infty)$ of any edge emanating from $v_0 \in \mathcal{T}$ is the Iwahori group (by contrast, edges in the cusps have non-trivial stabilizers) **[44]**. We conclude

$$\Delta_G = q^\delta + 1. \tag{4}$$

The first eigenvalue of the Laplace operator Since all stabilizers of vertices and edges in the finite graph $G := G_{\Gamma(Y,\mathfrak{n})}$ are trivial, its adjacency operator is the (unweighted) projection of the Hecke operator T_∞ on $\mathcal{T}$ corresponding to the characteristic function of the place ∞. The Ramanujan-Petersson conjecture for global function fields, proven by Drinfeld (**[17]**) implies that the eigenvalues of the operator T_∞ are bounded below by $2\sqrt{q^\delta}$.

Therefore, the operator

$$L'_G := (q^\delta + 1)\mathbf{1} - T_{\infty|G}$$

has non-zero eigenvalues

$$\lambda'_i \geq q^\delta + 1 - 2\sqrt{q^\delta}.$$

This operator is a perturbation of the Laplace operator L_G of G. More precisely, let B denote the diagonal matrix which has

$$B_v := \begin{cases} 1 & \text{if } v \text{ is adjacent to a cusp in } \Gamma(Y,\mathfrak{n})\backslash\mathcal{T}, \\ 0 & \text{otherwise;} \end{cases}$$

then

$$L_G + B = L'_G.$$

Now the Courant-Weyl inequalities (e.g., Theorem 2.1 in **[16]**) imply that λ_G is larger than the first eigenvalue of L'_G minus the largest eigenvalue of B, leading to

$$\lambda_G \geq q^\delta - 2\sqrt{q^\delta}. \tag{5}$$

Conclusion of the proof of the main bound. If we plug the collected data from equations (3), (4) and (5) in the lower bound from Theorem 1.3, we find

$$\mathrm{sgon}(X(Y,\mathfrak{n})) \geq \frac{q^\delta - 2\sqrt{q^\delta}}{2q(q^2-1)(q^\delta+2)} \cdot [\Gamma(Y) : \Gamma(Y,\mathfrak{n})],$$

as was to be proved.

Linear lower bound in the genus. We now show how to convert the lower bound on the gonality of X_Γ in terms of the index $[\Gamma(Y) : \Gamma]$ into a lower bound that is linear in the genus, of the form

$$\mathrm{gon}(X_\Gamma) \geq c_{K,\delta}(g(X_\Gamma) - 1),$$

for c_K a constant depending only on the ground field K and the degree δ of ∞. This is not entirely obvious in positive characteristic, due to wild ramification.

First of all, it is again enough to establish such a bound for a principal congruence subgroup $\Gamma(Y,\mathfrak{n})$. Indeed, from the Riemann-Hurwitz formula (see e.g. **[34]**) for the (Galois) cover (1) and formula (2), it follows that

$$\begin{aligned}
\mathrm{gon}(X_\Gamma) &\geq \frac{\mathrm{gon}(X(Y,\mathfrak{n}))}{[\Gamma : \Gamma(Y,\mathfrak{n})]} |\Gamma \cap Z| \\
&\geq \frac{c_{K,\delta}(g(X(Y,\mathfrak{n})) - 1)}{[\Gamma : \Gamma(Y,\mathfrak{n})]} |\Gamma \cap Z| \\
&\geq c_{K,\delta}(g(X_\Gamma) - 1 + \frac{r}{2}) \\
&\geq c_{K,\delta}(g(X_\Gamma) - 1),
\end{aligned}$$

where we have assumed that the desired bound holds for $\Gamma(Y,\mathfrak{n})$, and r is the ramification term in the Riemann-Hurwitz formula for the cover (1).

We now establish the bound for $X(Y,\mathfrak{n})$. If this curve has genus zero or one, the required bound for the gonality holds trivially. Therefore, we can assume $g(X(Y,\mathfrak{n})) \geq 2$. The Riemann-Hurwitz formula for the cover $X(Y,\mathfrak{n}) \to X(Y)$ implies a relation of the form

$$[\Gamma(Y) : \Gamma(Y,\mathfrak{n})] = (g(X(Y,\mathfrak{n})) - 1) \cdot \frac{2(q-1)}{2g(X(Y)) - 2 + R},$$

where R is the degree of the ramification divisor for this cover. Hence to prove our result, it suffices to prove a lower bound of the form

$$2g(X(Y)) - 2 + R \leq c'_{K,\delta}$$

for some constant $c'_{K,\delta}$ depending only on K and δ.

We recall some information about the ramification number R and the genus $g(X(Y))$ from **[22]** (There, the formulae are worked out for the principal component $Y = A \oplus A$ only, but hold in general). First of all, the genus of $X(Y)$ depends only on K and δ. Secondly, ramification takes place above elliptic points and cusps of $X(Y)$. Let us write $R = R_e + R_c$ with R_e the contribution from elliptic points, and R_c the contribution from cusps. The ramification above elliptic points is tame; and the number of elliptic points depends only on K and δ. Hence R_e is a constant in K and δ.

The ramification above the cusps is wild, but *weak*; this means that the second ramification groups are trivial, and the first ramification group is just the p-Sylow group of the stabilizer of the cusp (this follows, for example, from the fact that $X(Y)$ are Mumford curves, hence ordinary—since their Jacobian admits a Tate uniformization, and hence has maximal p-rank—, by applying a result of Nakajima **[34]**). In the end, we need a lower bound on

$$R_c = \frac{q^{d+1} - 2}{(q-1)q^d}$$

where $d = \deg(\mathfrak{n}) \geq 1$, that is independent of d; for example,

$$R_c \geq \frac{1}{q-1}$$

will do, and this finishes the proof. $\square$

8.9. Remark. In the "standard" case of a rational function field $K = \mathbf{F}_q(T)$ with a place ∞ of degree one, one can make all data explicit. The cover $X(\mathfrak{n}) \to X(1) \cong \mathbf{P}^1$ is ramified tamely at the unique elliptic point, of order $q+1$, and at the unique cusp, of order $q^d(q-1)$, where $d = \deg(\mathfrak{n})$. Hence the Riemann-Hurwitz formula becomes

$$\begin{aligned} 2g(X(\mathfrak{n}) - 1) &= [\Gamma(1) : \Gamma(\mathfrak{n})] \left(1 - \frac{1}{q+1} - \frac{1}{q^d(q-1)} - \frac{1}{q^d}\right) \\ &\leq [\Gamma(1) : \Gamma(\mathfrak{n})], \end{aligned}$$

and it follows that one can set $c_{K,\delta} = 2c_q$ in this case.

9. Modular degree of elliptic curves over function fields: proof of Theorem 1.5

9.1. Assume that K is a global function field, ∞ a place of K, and let E denote an elliptic curve over K with split multiplicative reduction at ∞ (every non-isotrivial curve acquires such a place of reduction after a finite extension of the ground field K). From the work of Drinfeld, it follows that E admits a *modular parametrization*

$$\phi \colon X_0(Y, \mathfrak{n}) \to E$$

(see Gekeler and Reversat **[25]**) for some suitable modular curves $X_0(Y, \mathfrak{n})$. This parametrization is defined over the maximal abelian extension H of K that is contained in the completion K_∞. One may study the (minimal) degree of such a modular parametrization, called the *modular degree*.

9.2. Remark. Contrary to the case of elliptic curves over $\mathbf{Q}$, in the case where $K = \mathbf{F}_q(T)$, Gekeler has proven that the modular degree always equals the congruence number of the associated automorphic form **[23] [13]**.

9.3. We first describe some of the structure of the modular curves $X_0(Y,\mathfrak{n})$. The scheme $M_0(\mathfrak{n})$, (coarsely) representing the moduli problem of rank-two Drinfeld modules with an $\mathfrak{n}$-isogeny, is defined over K, but is not absolutely irreducible if $\mathrm{Pic}(A)$ is non-trivial; it decomposes over $\mathbf{C}_\infty$ as

$$M_0(\mathfrak{n})(\mathbf{C}_\infty) = \bigsqcup_{Y\in\mathrm{Pic}(A)} \Gamma_0(Y,\mathfrak{n})\backslash\Omega,$$

where the components are defined over H, and sharply transitively permuted by the Galois group $\mathrm{Gal}(H/K)\cong\mathrm{Pic}(A)$. One may also describe the modular parametrizations for different Y simultaneously by a K-rational map $M_0(\mathfrak{n})\to E$, with $M_0(\mathfrak{n})$ not geometrically irreducible.

9.4. Since the elliptic curve E admits a map of degree two to $\mathbf{P}^1$, we find that

$$\mathrm{gon}(X_0(Y,\mathfrak{n}))\leq 2\deg(\phi).$$

Since we now have a lower bound

$$\mathrm{gon}(X_0(Y,\mathfrak{n}))\geq c_{q,\delta}[\Gamma(Y):\Gamma_0(Y,\mathfrak{n})],$$

we conclude that

$$\deg(\phi)\geq\frac{1}{2}c_{q,\delta}[\Gamma(Y):\Gamma_0(Y,\mathfrak{n})].$$

The desired result $\deg\phi\gg_{q,\delta}|\,\mathfrak{n}\,|_\infty$ follows from the following lemma.

9.5. Lemma. $[\Gamma(Y):\Gamma_0(Y,\mathfrak{n})]\geq|\,\mathfrak{n}\,|_\infty$.

Proof. Since both groups $\Gamma(Y)$ and $\Gamma_0(Y,\mathfrak{n})$ contain the center Z, this index is the degree of the covering $X_0(Y,\mathfrak{n})\to X(Y)$. Although the different components $X_0(Y,\mathfrak{n})$ of $M_0(\mathfrak{n})$ and $X(Y)$ of $M(1)$ depend on Y, they are Galois conjugate by $\mathrm{Gal}(H/K)\cong\mathrm{Pic}(A)$. Therefore, the covering *degree* of this cover does not depend on Y. Hence we can put $Y=A\oplus A$, and a standard computation then shows that there is a bijection

$$\mathrm{GL}(2,A)/\Gamma_0(\mathfrak{n})\xrightarrow{\sim}\mathbf{P}^1(A/\,\mathfrak{n}\,A)$$
$$\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\mapsto(a:c)$$

and hence

$$[\Gamma(Y):\Gamma_0(Y,\mathfrak{n})]=[\mathrm{GL}(2,A):\Gamma_0(\mathfrak{n})]=|\,\mathfrak{n}\,|_\infty\cdot\prod_{\mathfrak{p}|\mathfrak{n}}(1+|\,\mathfrak{p}\,|_\infty^{-1})\geq|\,\mathfrak{n}\,|_\infty,$$

as was to be proven. ☐

9.6. Corollary (= Theorem 1.5)**.** *Let E/K denote an elliptic curve with split multiplicative reduction at the place ∞, of conductor $\mathfrak{n}\cdot\infty$. Then the degree of a modular parametrization $\phi\colon X_0(Y,\mathfrak{n})\to E$ is bounded below by*

$$\deg\phi\geq\frac{1}{2}c_{q,\delta}[\Gamma(1):\Gamma_0(Y,\mathfrak{n})].$$

In particular, we have

$$\deg\phi\gg_{q,\delta}|\,\mathfrak{n}\,|_\infty.$$

9.7. Remark. For $K = \mathbf{F}_q(T)$ a rational function field and $\delta = 1$, Papikian **[36]**, has proven the degree conjecture (by estimating symmetric square L-functions): if E is an *optimal* Weil curve over $\mathbf{F}_q(T)$, then

$$\deg \phi \ll_{q,\varepsilon} |\,\mathfrak{n}\,|_\infty^{1+\varepsilon}$$

for any $\varepsilon > 0$. This combines with our lower bound.

9.8. Remark. Papikian **[37]** has also proven an upper bound on the modular degree of an optimal semistable elliptic curve for a general function field, depending on the Manin constant; one should note that in **[37]**, the bound is given without the Manin constant as a factor, since this was conjectured to always equal one, but Pál **[35]** has given examples where this is not the case, and he has also proven a general upper bound of the form

$$\deg(\phi) \ll_{K,\delta} |\,\mathfrak{n}\,|_\infty^2 \left(\log_q |\,\mathfrak{n}\,|_\infty\right)^3$$

for the degree of an optimal modular cover ϕ with squarefree conductor $\mathfrak{n}$, and such that the class number of K is coprime to the characteristic p.

10. Rational points of higher degree on Drinfeld modular curves: proof of Theorem 1.6

We first quote the positive characteristic analogue of a theorem of Frey **[21]**, that was proven by Schweizer **[42]**:

10.1. Proposition (**[42]**, Theorem 2.1)**.** *Let X denote a curve over K with a K-rational point, such that its Jacobian does not admit a $\overline{K}$-morphism to a curve defined over a finite field. If d is an integer such that $2d+1 \leq \mathrm{gon}_{\overline{K}}(X)$, then the set of points of degree d on X is finite, i.e.,*

$$\left| \bigcup_{[K':K]\leq d} X(K') \right| \leq \infty. \square$$

10.2. Remark. This result has now been improved into a quantitative statement by Cadoret and Tamagawa **[8]**, as follows: recall that gonality may alternatively be defined as the minimal d for which there exists a non-constant morphism from a $\mathbf{P}^1$ to the d-th symmetric power $X^{(d)}$ of the curve X. Define the *isogonality* $\mathrm{isogon}_K(X)$ of X as the minimal d for which there exists a non-constant morphism from a K-isotrivial curve to the d-th symmetric power $X^{(d)}$ of the curve X. Then the result from **[8]** says: *for any finitely generated field K of positive characteristic $p > 0$, and any smooth geometrically integral curve X over K, if d is a natural number with $2d+1 \leq \mathrm{gon}_{\overline{K}}(X)$ and $d+1 \leq \mathrm{isogon}_{\overline{K}}(X)$, then the set of points of degree $\leq d$ on X is finite.*

10.3. Theorem (= Theorem 1.6)**.** *If X_Γ is defined over a finite extension K_Γ of K, such that $X_\Gamma(K_\Gamma) \neq \emptyset$, then the set*

$$\bigcup_{[L:K_\Gamma]\leq \frac{1}{2}\left(c_{q,\delta}\cdot[\Gamma(1):\Gamma]-1\right)} X_\Gamma(L)$$

is finite.

Proof. First of all, we assume that $X_\Gamma(K_\Gamma) \neq \emptyset$. Secondly, the curves X_Γ are Mumford curves for the ∞-valuation, hence their Jacobian has split reduction at ∞, and hence it admits no map to an isotrivial curve. Therefore, the condition to apply Proposition 10.1 are satisfied by $X = X_\Gamma$ and $K = K_\Gamma$. □

10.4. Remark. Since Γ is a congruence group, the curve X_Γ is covered by some $X(Y, \mathfrak{n})$, and hence the curve X_Γ is defined over H and has H-rational points, namely, the cusps. Hence one may always choose $K_\Gamma = H$, but K_Γ might be chosen smaller.

We also deduce the following analogue of a result of Kamieny and Mazur **[26]**:

10.5. Corollary (= Theorem 1.7)**.** *If $\mathfrak{p}$ is a prime ideal in A, then the set of all rank two Drinfeld A-modules defined over some field extension L of K that satisfies the degree bound*

$$[LH : H] \leq \frac{1}{2} c_{q,\delta} \cdot |\,\mathfrak{p}\,|_\infty$$

that admit an L-rational $\mathfrak{p}$-isogeny is finite.

Proof. Recall that the scheme $M_0(\mathfrak{p})$, coarsely representing this moduli problem, decomposes over $\mathbf{C}_\infty$ as

$$M_0(\mathfrak{p})(\mathbf{C}_\infty) = \bigsqcup_{Y \in \mathrm{Pic}(A)} \Gamma_0(Y, \mathfrak{p}) \backslash \Omega,$$

where the components are defined over H, and all components have H-rational points, namely, the cusps.

Now a rank-two A-Drinfeld module ϕ over a field L with an L-rational $\mathfrak{p}$-isogeny gives rise to an L-rational point of $M_0(\mathfrak{p})$, and hence to an HL-rational point $[\phi] \in X_0(Y, \mathfrak{p})(HL)$ for some Y. Now the above theorem implies that

$$\bigcup_{[HL:H] \leq \frac{1}{2}(\mathrm{gon}_{\overline{H}}(X_0(Y,\mathfrak{p})-1)} X_0(Y, \mathfrak{p})(HL)$$

is finite. Now since by Theorem 1.4,

$$\mathrm{gon}_{\overline{H}}(X_0(Y, \mathfrak{p})) \geq c_{q,\delta}[\Gamma(Y) : \Gamma_0(Y, \mathfrak{p})],$$

and we have

$$[\Gamma(Y) : \Gamma_0(Y, \mathfrak{p})] = |\,\mathfrak{p}\,|_\infty + 1,$$

the result follows. □

10.6. Remark. Of course, the result is only interesting if $c_{q,\delta} > 0$, i.e., $q^\delta \geq 4$.

10.7. Corollary (= Corollary 1.8)**.** *Fix a prime $\mathfrak{p}$ of A. There is a uniform bound on the size of the $\mathfrak{p}$-primary torsion of any rank two A-Drinfeld module over L, where L ranges over all extensions for which the degree $[LH : H]$ is bounded by a given constant.*

Proof. The method of proof is similar to the one in Kamieny-Mazur **[26]**, as used in **[42]**, Thm. 2.4: the moduli space $M_0(\mathfrak{p}^e)$ has only finitely many LH-points as soon as

$$e \geq \log_q(2[LH : H]/c_{q,\delta}) / \log_q(|\,\mathfrak{p}\,|_\infty).$$

For each of the finitely many Drinfeld modules ϕ over LH corresponding to these points, Breuer **[7]** has shown that the open adelic image result of Pink and Rütsche **[38]** implies that the $\mathfrak{p}$-primary torsion $\phi[\mathfrak{p}^\infty]$ of ϕ is bounded by $C[LH : H]$, where C depends on ϕ, K and $\mathfrak{p}$. One may now maximize the bound as ϕ runs through these finitely many Drinfeld modules. Also, for any Drinfeld module ϕ, $|\phi[\mathfrak{p}^{e-1}]| \leq |\,\mathfrak{p}\,|_\infty^{2(e-1)}$. The result follows. □

10.8. Remark. In general, $[LH : H] \leq [L : K]$ (with equality if L and H are linearly disjoint). This shows that a bound of the form $[L : K] \leq d$ implies a bound of the form $[LH : H] \leq d$. Hence the uniform boundedness conjecture for rank-two A-Drinfeld modules over K **[40]** follows from the following statement: for fixed d, there are only finitely many $\mathfrak{p}$ such that there exists an L-rational $\mathfrak{p}$-torsion point on an A-Drinfeld module over L with $[L : K] \leq d$.

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